THE COLORED JONES POLYNOMIALS OF THE FIGURE-EIGHT KNOT AND ITS DEHN SURGERY SPACES

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ABSTRACT. We calculate limits of the colored Jones polynomials of the figure-eight knot and conclude that in most cases they determine the volumes and the Chern–Simons invariants of the three-manifolds obtained by Dehn surgeries along it.

1. INTRODUCTION

Let K be a knot and $J_N(K; t)$ be the colored Jones polynomial of K in the three-sphere S^3 corresponding to the N-dimensional representation of $sl_2(\mathbb{C})$ (see for example [4]). We normalize it so that $J_N(\text{unknot}; t) = 1$.

In [3], R. Kashaev introduced a series of numerical link invariants and proposed a conjecture that a limit of his invariants would determines the hyperbolic volume of the complement of a knot if it has a complete hyperbolic structure. On the other hand in [14] J. Murakami and the first author proved that Kashaev's invariant is equal to the absolute value of $J_N\big(K; \exp(2\pi\sqrt{-1}/N)\big)$. Moreover in [15] J. Murakami, M. Okamoto, T. Takata and the authors studies several knots and proposed the following complexification of Kashaev's conjecture.

Conjecture 1.1. *Let K be a hyperbolic knot (that is, the complement of K possesses a complete hyperbolic structure). Then*

$$2\pi \lim_{N\to\infty} \frac{\log J_N\big(K; \exp(2\pi\sqrt{-1}/N)\big)}{N} = \text{Vol}(S^3 \setminus K) + \sqrt{-1}\, \text{CS}(S^3 \setminus K),$$

where Vol *is the hyperbolic volume and* CS *is the Chern–Simons invariant for knots* [7].

See [10, 12, 17, 21, 19, 20, 22, 23] for related topics.

On the other hand the first author studied in [13] other limits of the colored Jones polynomials of the figure-eight knot and showed the following theorem.

Theorem 1.2. *Let E be the figure-eight knot and r a irrational number satisfying $5/6 < r < 7/6$ or $r = 1$. Then*

$$2\pi r \lim_{N\to\infty} \frac{\log\big|J_N\big(E; \exp(2\pi r\sqrt{-1}/N)\big)\big|}{N}$$

is the volume of the cone manifold with cone angle $2\pi|1 - r|$ whose singularity is the figure-eight knot.

He also calculated [11] 'fake' limits (optimistic limits) of the Witten–Reshetikhin–Turaev invariants of three-manifolds obtained from S^3 by integral Dehn surgeries along the figure-eight knot and observed they give the volumes and the Chern–Simons invariants of those manifolds.

The purpose of this paper is to show that limits of the colored Jones polynomials of the figure-eignt give the volume and the Chern–Simons invariants of the three-manifold obtained by general surgeries. To do that we study the asymptotic behavior of $J_N(E; \exp(2\pi r\sqrt{-1}/N))$ for large N by using the saddle point method with E the figure-eight knot. Then it turns out that it determines the potential function introduced by W. Neumann and D. Zagier [16] to study the volumes of Dehn surgered spaces along knots. (See also [1].) The we apply T. Yoshida's theorem [24] to show that the limit of $\log\left(J_N\left(E; \exp(2\pi r\sqrt{-1}/N)\right)\right)/N$ $(N \to \infty)$ gives the volume and the Chern–Simons invariant when $r \in \mathbb{C}$ is close to 1.

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2. Main result

To state our main result we recall some of the results in [16, 24].

Let K be a hyperbolic knot in the three-sphere S^3, that is, the complement $S^3 \setminus K$ has a complete hyperbolic structure. The complete hyperbolic structure can be deformed by a complex parameter u that is the logarithm of the ratio of the eigenvalues of the image of the meridian K by the holonomy representation $\pi_1(S^3 \setminus K) \to \mathrm{SL}_2(\mathbb{C})$. We also denote by v the logarithm of the eigenvalue of the image of the longitude. If the structure is complete, then $u = 0$. Let $\Phi(u)$ be the potential function introduced in [16, Theorem 3] that satisfies

$$\begin{cases} \dfrac{d\,\Phi(u)}{du} = 2v, \\ \Phi(0) = 0. \end{cases}$$

We also put

$$f(u) := \frac{\Phi(u)}{4} - \frac{uv}{4}.$$

(see [16, Equation (50)]).

Let ℓ be a loop on the boundary of the tubular neighborhood $N(K)$ of K in S^3. Suppose that the hyperbolic structure parameterized by u on $S^3 \setminus K$ can be completed to an orbifold K_u by attaching a solid torus such that its meridian coincides with ℓ. It is called generalized Dehn surgery and if $pu + qv = 2\pi\sqrt{-1}$ for coprime integers p and q, then it is usual (p, q)-Dehn surgery and K_u becomes a closed three-manifold [18, Chapter 4] and ℓ is homotopic to $p \cdot$meridian$+q \cdot$longitude in $\pi_1(\partial N(K))$. Let γ be the core of the solid torus. It is a geodesic in K_u, and its length and torsion are denoted by length(γ) and torsion(γ) respectively. (If one travels around γ then the plane perpendicular to it is rotated by torsion(γ). See [24, Definition 1.1] for the precise definition.) Then the volume and the Chern–Simons invariant of K_u can be obtained as follows ([24, Theorem 2], which was conjectured in [16, Conjecture]).

Theorem 2.1 (T. Yoshida)**.** *Put* $\lambda(\gamma) := \text{length}(\gamma) + \sqrt{-1}\,\text{torsion}(\gamma)$. *Then we have*

$$\text{Vol}(K_u) + \sqrt{-1}\,\text{CS}(K_u) \equiv \frac{f(u)}{\sqrt{-1}} - \frac{\pi}{2}\lambda(\gamma) + \text{Vol}\left(S^3 \setminus K\right) + \sqrt{-1}\,\text{CS}\left(S^3 \setminus K\right)$$

$$\text{mod } \pi^2\sqrt{-1}\mathbb{Z}.$$

Now we study the case where the knot is the figure-eight knot E.

Let u be a complex parameter near 0. Put

$$m := -\exp(u/2),$$

$$z := \frac{-m^4 + m^2 + 1 + \sqrt{(m^2 + m + 1)(m^2 + m - 1)(m^2 - m + 1)(m^2 - m - 1)}}{2m^2},$$

and

$$w := \frac{m^4 + m^2 - 1 + \sqrt{(m^2 + m + 1)(m^2 + m - 1)(m^2 - m + 1)(m^2 - m - 1)}}{2m^2}.$$

We take the branch of the square root so that $z = w = \exp(\pi/3)$ when $u = 0$. We also put

$$v := 2\log\left(z(1 - z)\right),$$

where the branch is chosen so that $v = 0$ when $u = 0$. Note that z and w satisfy the following equations.

$$\begin{cases} \log w + \log(1 - z) = u, \\ \log z + \log(1 - z) + \log w + \log(1 - w) = 0. \end{cases}$$

Put

$$f(u) := \frac{1}{2\pi}\left\{R(z) + R(w) - \frac{\pi^2}{6}\right\} - \frac{\sqrt{-1}}{2\pi}\,\text{Vol}\left(S^3 \setminus E\right)$$

and

$$\Phi(u) := 4f(u) + uv,$$

where

$$R(\xi) := \frac{1}{2}\log\xi\log(1 - \xi) - \int_0^\xi \frac{\log(1 - \eta)}{\eta}d\eta$$

is Roger's dilogarithm function. Then

$$\begin{cases} \dfrac{d\,\Phi(u)}{du} = 2v, \\ \Phi(0) = 0, \end{cases}$$

and

$$\text{Vol}(E_u) + \sqrt{-1}\,\text{CS}(E_u) \equiv \frac{f(u)}{\sqrt{-1}} - \frac{\pi}{2}\lambda(\gamma) + \text{Vol}\left(S^3 \setminus E\right) \quad \text{mod } \pi^2\sqrt{-1}\mathbb{Z},$$

since $\text{CS}\left(S^3 \setminus E\right) = 0$.

Now we state our main result.

Let $J_N(K; t)$ be the colored Jones polynomial of a knot K corresponding to the N-dimensional representation of $sl_2(\mathbb{C})$. We normalize it so that $J_N(\text{unknot}; t) = 1$. Then we have the following theorem.

Theorem 2.2. *Let E be the figure-eight knot. There exists a neighborhood U of 0 in $\mathbb{C}$ such that for any $u \in (U \setminus \pi\sqrt{-1}\mathbb{Q}) \cup \{0\}$, the following limit exists:*

$$(u + 2\pi\sqrt{-1}) \lim_{N \to \infty} \frac{\log\Big(J_N\big(E; \exp\big((u + 2\pi\sqrt{-1})/N\big)\big)\Big)}{N}.$$

Moreover if we denote the limit by $H(u)$, then it can be extended to a complex analytic function and it satisfies the following equalities.

$$\begin{cases} \dfrac{d\,H(u)}{du} = \dfrac{v}{2} + \pi\sqrt{-1}, \\[2mm] H(0) = \sqrt{-1}\,\mathrm{Vol}(S^3 \setminus E). \end{cases}$$

Remark 2.3. The precise meaning of the limit is as follows. If the ratio

$$\frac{J_N\big(E; \exp\big((u + 2\pi\sqrt{-1})/N\big)\big)}{\exp\big(N\,h(u)\big)}$$

grows polynomially with respect to N, then we denote the limit

$$\frac{\log\Big(J_N\big(E; \exp\big((u + 2\pi\sqrt{-1})/N\big)\big)\Big)}{N}$$

by $h(u)$. So it is defined modulo $2\pi\sqrt{-1}$.

As a corollary we can express the volume and the Chern–Simons invariant of a three-manifold obtained from the figure-eight knot by generalized Dehn surgery.

Corollary 2.4. *Let E_u be the three-manifold obtained from the figure-eight knot by generalized Dehn surgery corresponding to u near 0 such that $\pi u \notin \mathbb{Q}$. Then*

$$\mathrm{Vol}(E_u) + \sqrt{-1}\,\mathrm{CS}(E_u) \equiv \frac{H(u)}{\sqrt{-1}} - \pi u - \frac{uv}{4\sqrt{-1}} - \frac{\pi}{2}\lambda(\gamma) \mod \pi^2\sqrt{-1}\mathbb{Z}.$$

3. Limits of the colored Jones polynomials

K. Habiro [2] and T. Le showed that the colored Jones polynomial $J_N(E; t)$ of the figure-eight knot E is given as follows (see also [6]).

(3.1) $$J_N(E; t) = \sum_{n=0}^{N-1} \prod_{k=1}^{n} t^N \left(1 - t^{-N-k}\right)\left(1 - t^{-N+k}\right).$$

We will study the asymptotic behavior of $J_N(E; q_r)$ with $q_r := \exp(2\pi r\sqrt{-1}/N)$ for large N.

We approximate $\prod_{k=1}^{n}\left(1 - q_r^{-N\pm k}\right)$ by an integral. Note that

$$\prod_{k=1}^{n}\left(1 - q_r^{-N\pm k}\right) = \exp\left\{\frac{N}{2\pi}\sum_{k=1}^{n}\frac{2\pi}{N}\log\big(1 - \exp(\pm 2\pi k r\sqrt{-1}/N - 2\pi r\sqrt{-1})\big)\right\}.$$

Lemma 3.1. *If $r \notin \mathbb{R}$ then*

$$\log\left|1 - e^{2\pi b/N}\right| \leq \log\left|1 - \exp\left(sr\sqrt{-1} - 2\pi r\sqrt{-1}\right)\right| \leq \log\left(1 + e^{2\pi b}\right),$$

and

$$\log\left|1 - e^{2\pi b}\right| \leq \log\left|1 - \exp\left(-sr\sqrt{-1} - 2\pi r\sqrt{-1}\right)\right| < \log\left(1 + e^{4\pi|b|}\right)$$

where $r = a + b\sqrt{-1}$ with $a \in \mathbb{R}$, $0 \neq b \in \mathbb{R}$ and $0 \leq s \leq 2\pi - 2\pi/N$.

Proof. If we put $r := a + b\sqrt{-1}$ with $b \neq 0$, we have

$$\exp\left(sr\sqrt{-1} - 2\pi r\sqrt{-1}\right) = e^{-bs+2\pi b} \exp\left((as - 2\pi a)\sqrt{-1}\right).$$

Therefore $\left|1 - \exp\left(sr\sqrt{-1} - 2\pi r\sqrt{-1}\right)\right|$ is smaller (greater, respectively) than or equal to the longest (shortest, respectively) distance between $1 \in \mathbb{C}$ and the circle centered at 0 with radius $e^{-bs+2\pi b}$. So if $b > 0$,

$$1 + e^{2\pi b} \geq \left|1 - \exp\left(sr\sqrt{-1} - 2\pi r\sqrt{-1}\right)\right| \geq e^{-b(2\pi-2\pi/N)+2\pi b} - 1 = e^{2\pi b/N} - 1$$

and if $b < 0$,

$$1 + e^{2\pi b} \geq \left|1 - \exp\left(sr\sqrt{-1} - 2\pi r\sqrt{-1}\right)\right| \geq 1 - e^{2\pi b/N}.$$

Similarly if $b > 0$,

$$e^{2\pi b} - 1 \leq \left|1 - \exp\left(-sr\sqrt{-1} - 2\pi r\sqrt{-1}\right)\right| \leq e^{b(2\pi-2\pi/N)+2\pi b} + 1 < e^{4\pi b} + 1$$

and if $b < 0$,

$$1 - e^{2\pi b} \leq \left|1 - \exp\left(-sr\sqrt{-1} - 2\pi r\sqrt{-1}\right)\right| \leq e^{4\pi b - 2\pi b/N} + 1 \leq e^{2\pi b} + 1.$$

$\square$

We put

$$\varphi_{N,\pm}(n) := \sum_{k=1}^{n} \frac{2\pi}{N} \log\left(1 - \exp(\pm 2\pi kr\sqrt{-1}/N - 2\pi r\sqrt{-1})\right)$$

$$- \int_{0}^{2\pi n/N} \log\left(1 - \exp(\pm sr\sqrt{-1} - 2\pi r\sqrt{-1})\right) ds.$$

Lemma 3.2. *If $r \notin \mathbb{R}$, then there exist constants $d \geq 1$ and $\delta > 0$ such that*

$$\left|\operatorname{Re}\varphi_{N,+}(n)\right| < \frac{2\pi d}{N} \left\{\log\left(1 + e^{2\pi b}\right) - \log\left(\frac{2\pi |b|\delta}{N}\right)\right\}$$

and

$$\left|\operatorname{Re}\varphi_{N,-}(n)\right| < \frac{2\pi d}{N} \left\{\log\left(1 + e^{4\pi |b|}\right) - \log\left|1 - e^{2\pi b}\right|\right\}$$

for sufficiently large N.

Proof. First we note that if a real continuous function $g(s)$ $(0 \leq s \leq A)$ has c extremes, then

$$\left|\int_{0}^{nA/N} g(s)\,ds - \sum_{k=1}^{n} \frac{A}{N} g(kA/N)\right| \leq (2c+1)\frac{A}{N} \left(\max_{0 \leq s \leq A} g(s) - \min_{0 \leq s \leq A} g(s)\right).$$

This is because if g is monotonic in $[jA/N, (j+l)A/N]$

$$\left|\int_{jA/N}^{(j+l)A/N} g(s)\,ds - \sum_{k=j+1}^{j+l} \frac{A}{N} g(kA/N)\right|$$

$$\leq \frac{A}{N}\left|g(jA/N) - g((j+l)A/N)\right| \leq \frac{A}{N}\left(\max_{0 \leq s \leq A} g(s) - \min_{0 \leq s \leq A} g(s)\right)$$

and there are $c + 1$ such intervals. For other c intervals $[jA/N, (j+1)A/N]$ clearly

$$\left|\int_{jA/N}^{(j+1)A/N} g(s)\,ds - \frac{A}{N} g((j+1)A/N)\right| \leq \frac{A}{N}\left(\max_{0 \leq s \leq A} g(s) - \min_{0 \leq s \leq A} g(s)\right)$$

holds.

Therefore from Lemma 3.1 there exists a constant d such that

$$|\text{Re}\,\varphi_{N,+}(n)| \leq \frac{2\pi d}{N}\left\{\log\left(1 + e^{2\pi b}\right) - \log\left|1 - e^{2\pi b/N}\right|\right\}$$

and

$$|\text{Re}\,\varphi_{N,-}(n)| < \frac{2\pi d}{N}\left\{\log\left(1 + e^{4\pi|b|}\right) - \log\left|1 - e^{2\pi b}\right|\right\}$$

Since if $b > 0$,

$$e^{2\pi b/N} - 1 = \frac{2\pi b}{N} + \frac{(2\pi b)^2}{2N^2} + \cdots > \frac{2\pi b}{N}$$

for any N and if $b < 0$,

$$1 - e^{2\pi b/N} = -\frac{2\pi b}{N} - \frac{(2\pi b)^2}{2N^2} - \cdots > -\frac{2\pi b\delta}{N}$$

for any $0 < \delta < 1$ if N is sufficiently large. This completes the proof. $\qquad\square$

Therefore we have

$$D^{-1}N^{-d} < \left|\exp\left(\frac{N}{2\pi}\varphi_{N,\pm}(n)\right)\right| < DN^d$$

for some constants $D > 0$ and $d \geq 1$ for sufficient large N.

We also have

Lemma 3.3. *If $r \notin \mathbb{R}$, then for any positive number ε*

$$|\text{Im}\,\varphi_{N,\pm}(n)| < \varepsilon$$

for sufficiently large N.

Proof. Clearly

$$\max_{2\pi/N \leq s \leq 2\pi - 2\pi/N}\arg\left(1 - \exp(\pm sr\sqrt{-1} - 2\pi r\sqrt{-1})\right)$$

$$- \min_{2\pi/N \leq s \leq 2\pi - 2\pi/N}\arg\left(1 - \exp(\pm sr\sqrt{-1} - 2\pi r\sqrt{-1})\right)$$

is bounded by a positive constant depending only r. So we can show as in the previous lemma that

$$|\text{Im}\,\varphi_{N,\pm}(n)| = \left|\sum_{k=1}^{n}\frac{2\pi}{N}\arg\left(1 - \exp(\pm sr\sqrt{-1} - 2\pi r\sqrt{-1})\right)\right.$$

$$\left. - \int_{0}^{2\pi n/N}\arg\left(1 - \exp(\pm sr\sqrt{-1} - 2\pi r\sqrt{-1})\right)\right|$$

is bounded by a positive constant times $1/N$ and the result follows. $\qquad\square$

Putting

$$\chi_{N,\pm}(n) := \exp\left(\frac{N}{2\pi}\varphi_{N,\pm}(n)\right),$$

we have

$$\prod_{k=1}^{n}\left(1 - q_r^{-N\pm k}\right)$$

$$= \chi_{N,\pm}(n)\exp\left\{\frac{N}{2\pi}\int_{0}^{2\pi n/N}\log\left(1 - \exp(\pm sr\sqrt{-1} - 2\pi r\sqrt{-1})\right)ds\right\}$$

with $D^{-1}N^{-d} < |\chi_{N,\pm}(n)| < DN^d$ and $|\arg \chi_{N,\pm(n)}| < \varepsilon$ if $r \notin \mathbb{R}$. Therefore

$$\prod_{k=1}^{n} \left(1 - q_r^{-N\pm k}\right)$$

$$= \chi_{N,\pm}(n) \exp\left\{ \frac{\pm N}{2\pi r\sqrt{-1}} \int_{\exp(-2\pi r\sqrt{-1})}^{\exp(\pm 2\pi n r\sqrt{-1}/N - 2\pi r\sqrt{-1})} \frac{\log(1-u)}{u}\, du \right\}$$

$$= \chi_{N,\pm}(n) \exp\left\{ \frac{\pm N}{2\pi r\sqrt{-1}} \left(\mathrm{Li}_2\left(m^{-2}\right) - \mathrm{Li}_2\left(q_r^{\pm n} m^{-2}\right) \right) \right\}.$$

Here we put

$$m := \exp(\pi r \sqrt{-1})$$

and

$$\mathrm{Li}_2(z) := -\int_0^z \frac{\log(1-u)}{u}\, du$$

is the dilogarithm function.

Thus we have

$$J_N(E; q_r)$$

$$= \sum_{n=0}^{N-1} q_r^{Nn} \prod_{k=1}^{n} \left(1 - q_r^{-N+k}\right)\left(1 - q_r^{-N-k}\right)$$

$$= \sum_{n=0}^{N-1} q_r^{Nn} \chi_{N,+}(n) \exp\left\{ \frac{N}{2\pi r\sqrt{-1}} \left(\mathrm{Li}_2\left(m^{-2}\right) - \mathrm{Li}_2\left(q_r^n m^{-2}\right) \right) \right\}$$

$$\times \chi_{N,-}(n) \exp\left\{ \frac{-N}{2\pi r\sqrt{-1}} \left(\mathrm{Li}_2\left(m^{-2}\right) - \mathrm{Li}_2\left(q_r^{-n} m^{-2}\right) \right) \right\}$$

$$= \sum_{n=0}^{N-1} \chi_N(n) \exp\left\{ \frac{N}{2\pi r\sqrt{-1}} \left(\mathrm{Li}_2\left(q_r^{-n} m^{-2}\right) - \mathrm{Li}_2\left(q_r^n m^{-2}\right) \right. \right.$$

$$\left. \left. + \left(\log\left(-q_r^n\right) + \pi\sqrt{-1}\right)\left(\log m^2\right) \right) \right\}$$

$$= \sum_{n=0}^{N-1} \chi_N(n) \exp\left\{ \frac{N}{2\pi r\sqrt{-1}} H\left(q_r^n, m^2\right) \right\},$$

where we put

$$H(\xi, \eta) := \mathrm{Li}_2\left(\xi^{-1}\eta^{-1}\right) - \mathrm{Li}_2\left(\xi\eta^{-1}\right) + \left(\log(-\xi) + \pi\sqrt{-1}\right)\log\eta$$

and $\chi_N(n) := \chi_{N,+}(n)\chi_{N,-}(n)$. Note that $D^{-2}N^{-2d} < |\chi(n)| < D^2 N^{2d}$. Note also that we use $\log(-\xi) + \pi\sqrt{-1}$ instead of $\log\xi$ since we choose the branch of log as $(0, +\infty)$ and that of Li_2 as $(1, +\infty)$.

Let $\Psi_N(z)$ be an analytic function such that $\Psi_N\left(\exp(2\pi n\sqrt{-1}/N)\right) = \chi_N(n)$, that $P(N^{-1}) < |\Psi_N(z)| < Q(N)$ with some polynomials P and Q, and that $|\arg \Psi_N(z)| < \varepsilon$ near C defined below. Since the residue around $z = \exp(2\pi n\sqrt{-1}/N)$ of $1/\left(z(1 - z^{-N})\right)$ is $1/N$, we have from the residue theorem

$$J_N(E; q_r) = 1 + \frac{N}{2\pi\sqrt{-1}} \int_C \frac{\Psi_N(z)}{z(1 - z^{-N})} \exp\left(\frac{N}{2\pi r\sqrt{-1}} H(z^r, m^2) \right) dz,$$

where $C = C_+ \cup C_-$ is defined by

$$
\begin{cases}
C_+ & := \{z \in \mathbb{C} \mid |z| = 1 + \kappa, \pi/N \leq \arg z \leq 2\pi - \pi/N\} \\
& \quad \cup \{t \exp(\sqrt{-1}/N) \in \mathbb{C} \mid 1 - \kappa \leq t \leq 1 + \kappa\} \\
& \quad \cup \{t \exp(-\sqrt{-1}/N) \in \mathbb{C} \mid 1 - \kappa \leq t \leq 1 + \kappa\} \\
C_- & := \{z \in \mathbb{C} \mid |z| = 1 - \kappa, \pi/N \leq \arg z \leq 2\pi - \pi/N\}
\end{cases}
$$

for small $\kappa > 0$. Now we have

$$
J_N(E; q_r)
$$

$$
= 1 + \frac{N}{2\pi\sqrt{-1}} \int_{C_+} \frac{\Psi_N(z)}{z} \exp\left(\frac{N}{2\pi r\sqrt{-1}} H(z^r, m^2)\right) dz
$$

$$
+ \frac{N}{2\pi\sqrt{-1}} \int_{C_+} \frac{\Psi_N(z)}{z(1 - z^{-N})} \exp\left(\frac{N}{2\pi r\sqrt{-1}} \left\{H(z^r, m^2) - 2\pi\sqrt{-1}\log z^r\right\}\right) dz
$$

$$
+ \frac{N}{2\pi\sqrt{-1}} \int_{C_-} \frac{\Psi_N(z)}{z(z^N - 1)} \exp\left(\frac{N}{2\pi r\sqrt{-1}} \left\{H(z^r, m^2) + 2\pi\sqrt{-1}\log z^r\right\}\right) dz.
$$

Here we want to apply the steepest descent method (or the saddle point method, see for example [5, Theorem 7.2.9])

Theorem 3.4 (Steepest Descent Method). *Let* $\Gamma \colon [a, b] \to \mathbb{C}$ *be a* C^1 *curve. Let* $h(z)$ *be a continuous function on* Γ *that is analytic at* $z_0 = \Gamma(t_0)$, *and* $g_N(z)$ *be a continuous function on* $z \in \Gamma$ *for each* $N = 1, 2, 3, \ldots$ *with* $g_N(z_0) \neq 0$ *such that* $|g_N(z)| < N^p$ *for some* $p > 0$. *We also assume the following conditions: For sufficient large integer* N,

(1) $\int_\Gamma g_N(z) \exp(Nh(z)) dz$ *converges absolutely.*
(2) $d\,h(z_0)/dz = 0$ *and* $d^2\,h(z_0)/dz^2 \neq 0$.
(3) $\mathrm{Im}(Nh(z))$ *is constant for* z *on* c *in some neighborhood of* z_0.
(4) $\mathrm{Re}(Nh(z))$ *takes its strict maximum along* Γ *at* z_0.

Then

$$
\int_\Gamma g_N(z) \exp(Nh(z)) dz \underset{N \to \infty}{\sim} \frac{\sqrt{2\pi} g_N(z_0) \exp(Nh(z_0))}{\sqrt{N}\sqrt{-d^2\,h(z_0)/dz^2}}
$$

for appropriate chosen square roots. Here $\underset{N \to \infty}{\sim}$ *means the ratio of both hand sides goes to* 1 *when* $N \to \infty$.

Remark 3.5. In [5, Theorem 7.2.9] it is assumed that $g_N(z)$ does not depend on N but the result is also true if we assume that $|g_N(z)|$ grows polynomially. See Exercises 8 and 10 in Section 7.3 of [5].

Putting $h(z) := \frac{1}{2\pi r\sqrt{-1}} H(z^r, m^2)$, $g(z) := \Psi_N(z)/z$, we can prove the following proposition.

Proposition 3.6. *We have*

$$
\int_{C_+} \frac{\Psi_N(z)}{z} \exp\left(\frac{N}{2\pi r\sqrt{-1}} H(z^r, m^2)\right) dz
$$

$$
\underset{N \to \infty}{\sim} \frac{\sqrt{2\pi} \Psi_N(y^{1/r})}{r\sqrt{N}\sqrt{y - y^{-1}}} \exp\left(\frac{N}{2\pi r\sqrt{-1}} H(y, m^2)\right)
$$

with

$$
y := \frac{m^4 - m^2 + 1 - \sqrt{(m^2 + m + 1)(m^2 + m - 1)(m^2 - m + 1)(m^2 - m - 1)}}{2m^2},
$$

where the branch of the square root in the definition of y is taken so that if $m = -1$, then $y = \exp(-\pi\sqrt{-1}/3)$. Note that y satisfies the following equality:

$$(3.2) \qquad\qquad y + y^{-1} = m^2 - 1 + m^{-2}.$$

Proof. First of all we calculate derivatives of $H(z^r, m^2)$. We have

$$\begin{aligned}
\frac{\partial\, H(z^r, m^2)}{\partial z} &= \frac{\partial\, H(\xi, \eta)}{\partial \xi}\frac{d\, z^r}{dz}\\
&= \frac{rz^{r-1}}{z^r}\left\{\log(1 - z^{-r}m^{-2}) + \log(1 - z^r m^{-2}) + \log m^2\right\}\\
&= \frac{r}{z}\log\left(m^2 - (z^r + z^{-r}) + m^{-2}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial^2\, H(z^r, m^2)}{\partial z^2}\\
&\quad= -\frac{r}{z^2}\log\left(m^2 - (z^r + z^{-r}) + m^{-2}\right) + \frac{r}{z}\times\frac{-rz^{r-1} + rz^{-r-1}}{m^2 - (z^r + z^{-r} + m^{-2})}.
\end{aligned}$$

Therefore $y^{1/r}$ gives a solution to $\partial\, H(z^r, m^2)/\partial z = 0$, and

$$\left.\frac{\partial^2\, H(z^r, m^2)}{\partial z^2}\right|_{z = y^{1/r}} = r^2 y^{-2/r}(-y + y^{-1}).$$

Now we will consider the case when $r = 1$ precisely. In this case $y = \exp(-\pi\sqrt{-1}/3)$. We change C_+ to C'_+ where

$$\begin{aligned}
C'_+ := {}&\{z \in \mathbb{C} \mid |z| = 1, \pi/N \leq \arg z \leq 2\pi - \pi/N\}\\
&\cup \left\{t\exp(\sqrt{-1}/N) \in \mathbb{C} \mid 1 - \kappa \leq t \leq 1\right\}\\
&\cup \left\{t\exp(-\sqrt{-1}/N) \in \mathbb{C} \mid 1 - \kappa \leq t \leq 1\right\}.
\end{aligned}$$

Then on the unit circle

$$\begin{aligned}
H(\exp(\theta\sqrt{-1}), 1) &= \operatorname{Li}_2(\exp(-\theta\sqrt{-1})) - \operatorname{Li}_2(\exp(\theta\sqrt{-1}))\\
&= \left\{\frac{\pi^2}{6} - \frac{2\pi - \theta}{2}\left(\pi - \frac{2\pi - \theta}{2}\right) + 2\sqrt{-1}\,\text{Л}\left(\frac{2\pi - \theta}{2}\right)\right\}\\
(3.3) &\quad - \left\{\frac{\pi^2}{6} - \frac{\theta}{2}\left(\pi - \frac{\theta}{2}\right) + 2\sqrt{-1}\,\text{Л}\left(\frac{\theta}{2}\right)\right\}\\
&= -4\sqrt{-1}\,\text{Л}\left(\frac{\theta}{2}\right).
\end{aligned}$$

Here we use the following formula:

$$(3.4) \qquad \operatorname{Li}_2\left(\exp(\beta\sqrt{-1})\right) = \frac{\pi^2}{6} - \frac{\beta}{2}\left(\pi - \frac{\beta}{2}\right) + 2\sqrt{-1}\,\text{Л}\left(\frac{\beta}{2}\right)$$

for $0 \leq \beta \leq 2\pi$ and the fact that Л is an odd function with period π. (See for example [9, p. 18].) So on C'_+ $\operatorname{Im} H(z, 1)$ takes its maximum at y and near y $\operatorname{Re} H(z, 1) = 0$. Moreover in this case

$$\left.\frac{\partial^2\, H(z, 1)}{\partial z^2}\right|_{z = y} = \frac{-3 - \sqrt{3}\sqrt{-1}}{2} \neq 0.$$

Therefore we can apply the saddle point method and the proposition follows when $r = 1$.

If r is near 1 we can also apply the saddle point method similarly from the continuity of $H(z^r, m^2)$ with respect to r, completing the proof. $\qquad\square$

From Lemma 3.7 below we see

$$\left| \int_{C_+} \frac{\Psi_N(z)}{z(1-z^{-N})} \exp\left(\frac{N}{2\pi r\sqrt{-1}} \left\{ H(z^r, m^2) - 2\pi\sqrt{-1}\log z^r \right\} \right) dz \right|$$

$$\leq \int_{C_+} \frac{|\Psi_N(z)|}{|z(1-z^{-N})|} \exp\left(\frac{N}{2\pi} \operatorname{Im}\left(\frac{H(z^r, m^2) - 2\pi\sqrt{-1}\log z^r}{r} \right) \right) |dz|$$

$$< \exp\left(\frac{N}{2\pi} \operatorname{Im}\left(\frac{H(y, m^2)}{r} \right) \right) \int_{C_+} \frac{|\Psi_N(z)|}{|z(1-z^{-N})|} |dz|$$

$$\leq \exp\left(\frac{N}{2\pi} \operatorname{Im}\left(\frac{H(y, m^2)}{r} \right) \right) \frac{|C_+|Q(N)}{(1+\kappa)\left(1 - (1+\kappa)^{-N}\right)}$$

and

$$\left| \int_{C_-} \frac{\Psi_N(z)}{z(z^N-1)} \exp\left(\frac{N}{2\pi r\sqrt{-1}} \left\{ H(z^r, m^2) + 2\pi\sqrt{-1}\log z^r \right\} \right) dz \right|$$

$$< \exp\left(\frac{N}{2\pi} \operatorname{Im}\left(\frac{H(y, m^2)}{r} \right) \right) \frac{|C_-|Q(N)}{(1-\kappa)\left(1 - (1-\kappa)^N\right)},$$

where $|C_\pm|$ is the length of $C_\pm$. Now we have

(3.5)

$$\log\big(J_N(E; q_r)\big)/N - \frac{H(y, m^2)}{2\pi r\sqrt{-1}} - \log\left(\frac{\sqrt{2\pi}\Psi_N(y^{1/r})}{r\sqrt{N}\sqrt{y - y^{-1}}} \right) \Big/ N$$

$$= \log\left(\frac{r\sqrt{N}\sqrt{y - y^{-1}}}{\sqrt{2\pi}\Psi_N(y^{1/r})} \exp\left(\frac{-N}{2\pi r\sqrt{-1}} H(y, m^2) \right) \right.$$

$$+ \frac{N}{2\pi\sqrt{-1}} \frac{\int_{C_+} \frac{\Psi_N(z)}{z} \exp\left(\frac{N}{2\pi r\sqrt{-1}} H(z^r, m^2) \right) dz}{\frac{\sqrt{2\pi}\Psi_N(y^{1/r})}{r\sqrt{N}\sqrt{y-y^{-1}}} \exp\left(\frac{N}{2\pi r\sqrt{-1}} H(y, m^2) \right)}$$

$$+ \frac{N}{2\pi\sqrt{-1}} \frac{\int_{C_+} \frac{\Psi_N(z)}{z(1-z^{-N})} \exp\left(\frac{N}{2\pi r\sqrt{-1}} H(z^r, m^2) - 2\pi\sqrt{-1}\log z^r \right) dz}{\frac{\sqrt{2\pi}\Psi_N(y^{1/r})}{r\sqrt{N}\sqrt{y-y^{-1}}} \exp\left(\frac{N}{2\pi r\sqrt{-1}} H(y, m^2) \right)}$$

$$+ \left. \frac{N}{2\pi\sqrt{-1}} \frac{\int_{C_-} \frac{\Psi_N(z)}{z(z^N-1)} \exp\left(\frac{N}{2\pi r\sqrt{-1}} H(z^r, m^2) + 2\pi\sqrt{-1}\log z^r \right) dz}{\frac{\sqrt{2\pi}\Psi_N(y^{1/r})}{r\sqrt{N}\sqrt{y-y^{-1}}} \exp\left(\frac{N}{2\pi r\sqrt{-1}} H(y, m^2) \right)} \right) \Big/ N.$$

We will estimate each term in log. First we observe that

$$\left| \frac{r\sqrt{N}\sqrt{y - y^{-1}}}{\sqrt{2\pi}\Psi_N(y^{1/r})} \exp\left(\frac{-N}{2\pi r\sqrt{-1}} H(y, m^2) \right) \right|$$

$$= \frac{|r|\sqrt{N}|\sqrt{y - y^{-1}}|}{\sqrt{2\pi}\,|\Psi_N(y^{1/r})|} \exp\left(-N \operatorname{Re}\left(\frac{H(y, m^2)}{2\pi r\sqrt{-1}} \right) \right)$$

goes to 0. This is because $P(N^{-1}) < |\Psi_N(y^{1/r})| < Q(N)$ for polynomials P and Q, and $\operatorname{Re}\big(H(y, m^2)/2\pi r\sqrt{-1}\big) > 0$ if r is near 1 since when $r = 1$

$$\frac{H(y, m^2)}{2\pi r\sqrt{-1}} = \frac{H(\exp(-\pi\sqrt{-1}/3), 1)}{2\pi\sqrt{-1}} = \frac{2\,\text{Л}(\pi/6)}{\pi} > 0$$

from (3.3). Next we have

$$\lim_{N\to\infty} \frac{\int_{C_+} \frac{\Psi_N(z)}{z} \exp\left(\frac{N}{2\pi r\sqrt{-1}} H(z^r, m^2)\right) dz}{\frac{\sqrt{2\pi}\Psi_N(y^{1/r})}{r\sqrt{N}\sqrt{y-y^{-1}}} \exp\left(\frac{N}{2\pi r\sqrt{-1}} H(y, m^2)\right)} = 1,$$

$$\left| \frac{N}{2\pi\sqrt{-1}} \frac{\int_{C_+} \frac{\Psi_N(z)}{z(1-z^{-N})} \exp\left(\frac{N}{2\pi r\sqrt{-1}} H(z^r, m^2) - 2\pi\sqrt{-1}\log z^r\right) dz}{\frac{\sqrt{2\pi}\Psi_N(y^{1/r})}{r\sqrt{N}\sqrt{y-y^{-1}}} \exp\left(\frac{N}{2\pi r\sqrt{-1}} H(y, m^2)\right)} \right|$$

$$< \frac{|rN^{3/2}\sqrt{y-y^{-1}}||C_+|Q(N)}{(2\pi)^{3/2}|\Psi_N(y^{1/r})|(1+\kappa)\left(1-(1+\kappa)^{-N}\right)}$$

$$< \frac{|rN^{3/2}\sqrt{y-y^{-1}}||C_+|Q(N)(1+N\kappa)}{(2\pi)^{3/2}P(N^{-1}))(1+\kappa)N\kappa}$$

and

$$\left| \frac{N}{2\pi\sqrt{-1}} \frac{\int_{C_-} \frac{\Psi_N(z)}{z(z^N-1)} \exp\left(\frac{N}{2\pi r\sqrt{-1}} H(z^r, m^2) + 2\pi\sqrt{-1}\log z^r\right) dz}{\frac{\sqrt{2\pi}\Psi_N(y^{1/r})}{r\sqrt{N}\sqrt{y-y^{-1}}} \exp\left(\frac{N}{2\pi r\sqrt{-1}} H(y, m^2)\right)} \right|$$

$$< \frac{|rN^{3/2}\sqrt{y-y^{-1}}||C_-|Q(N)}{(2\pi)^{3/2}|\Psi_N(y^{1/r})|(1-\kappa)\left(1-(1-\kappa)^N\right)}$$

$$< \frac{|rN^{3/2}\sqrt{y-y^{-1}}||C_-|Q(N)}{(2\pi)^{3/2}P(N^{-2})(1-\kappa)(\kappa+(N-1)\kappa^2/2)}.$$

So we see that the absolute values of the first, third, and fourth terms in log in (3.5) are smaller than a polynomial in N. Therefore (3.5) goes to 0 when $N \to \infty$ and so we have

$$\lim_{N\to\infty} \frac{\log\big(J_N(E; q_r)\big)}{N}$$

$$= \frac{H(y, m^2)}{2\pi r\sqrt{-1}} + \lim_{N\to\infty} \log\left(\frac{\sqrt{2\pi}\Psi_N(y^{1/r})}{r\sqrt{N}\sqrt{y-y^{-1}}}\right) \Big/ N$$

$$= \frac{H(y, m^2)}{2\pi r\sqrt{-1}} + \lim_{N\to\infty} \frac{\log\big|\Psi_N(y^{1/r})\big|}{N} + \sqrt{-1} \lim_{N\to\infty} \frac{\arg \Psi_N(y^{1/r})}{N}$$

$$= \frac{H(y, m^2)}{2\pi r\sqrt{-1}}$$

since $P(N^{-1}) < |\Psi_N(y^{1/r})| < Q(N)$ and $|\arg \Psi_N(y^{1/r})| < \varepsilon$.

Lemma 3.7. *There exists a neighborhood U' of $1 \in \mathbb{C}$ such that if $r \in U'$,*

$$(3.6) \qquad \mathrm{Im}\left(\frac{H\left(z^r, m^2\right) \mp 2\pi\sqrt{-1}\log z^r}{r}\right) < \mathrm{Im}\left(\frac{H(y, m^2)}{r}\right)$$

for $z \in C_\pm$.

Proof. We will show that the inequality holds when $r = 1$. Then from the compactness of $C_\pm$ and the continuity we see that there exists such a neighborhood U' of 1.

Put

$$h_\mp(z) := H(z, 1) \mp 2\pi\sqrt{-1}\log z - H\big(\exp(-\pi\sqrt{-1}/3), 1\big)$$

$$= \mathrm{Li}_2(1/z) - \mathrm{Li}_2(z) \mp 2\pi\sqrt{-1}\log z - \mathrm{Li}_2\left(e^{\pi\sqrt{-1}/3}\right) + \mathrm{Li}_2\left(e^{-\pi\sqrt{-1}/3}\right).$$

and we regard it as a function of ρ and θ with $z = \rho\exp(\theta\sqrt{-1})$; that is

$$h_{\mp}(\rho,\theta) = \mathrm{Li}_2\left(\rho^{-1}e^{-\theta\sqrt{-1}}\right) - \mathrm{Li}_2\left(\rho e^{\theta\sqrt{-1}}\right) \pm 2\pi\theta \mp 2\pi\sqrt{-1}\log\rho$$
$$- \mathrm{Li}_2\left(e^{\pi\sqrt{-1}/3}\right) + \mathrm{Li}_2\left(e^{-\pi\sqrt{-1}/3}\right).$$

We also put

$$\begin{cases} D_+ := \{z \in \mathbb{C} \mid |z| \geq 1\}, \\ D_- := \{z \in \mathbb{C} \mid |z| \leq 1\}. \end{cases}$$

We will show that $\mathrm{Im}\, h_{\mp}(\rho,\theta)$ on $D_{\pm}$ takes its unique maximum 0 at $\exp(-\pi\sqrt{-1}/3)$.

Since

$$\frac{\partial\, h_{\mp}(\rho,\theta)}{\partial\theta} = \sqrt{-1}\log\left(1 - \rho^{-1}e^{-\theta\sqrt{-1}}\right) + \sqrt{-1}\log\left(1 - \rho e^{\theta\sqrt{-1}}\right) \pm 2\pi,$$

we have

$$\frac{\partial\, \mathrm{Im}\, h_{\mp}(\rho,\theta)}{\partial\,\theta} = \log\left|\left(1 - \rho^{-1}e^{-\theta\sqrt{-1}}\right)\left(1 - \rho e^{\theta\sqrt{-1}}\right)\right|.$$

Since

$$\left|\left(1 - \rho^{-1}e^{-\theta\sqrt{-1}}\right)\left(1 - \rho e^{\theta\sqrt{-1}}\right)\right|^2 = \left(1 + \rho^{-2} - 2\rho^{-1}\cos\theta\right)\left(1 + \rho^2 - 2\rho\cos\theta\right)$$
$$= \left(\rho + \rho^{-1} - 2\cos\theta\right)^2,$$

$\mathrm{Im}\, h_{\mp}(\rho,\theta)$ takes its maximum at $\theta = 2\pi - \arccos\left(\dfrac{\rho + \rho^{-1} - 1}{2}\right)$ for a fixed ρ if $\left(3 - \sqrt{5}\right)/2 < \rho < \left(3 + \sqrt{5}\right)/2$.

Now we put $\theta(\rho) := 2\pi - \arccos\left(\dfrac{\rho + \rho^{-1} - 1}{2}\right)$ and denote $h_{\mp}(\rho,\theta(\rho))$ by $h_{\mp}(\rho)$.

Since

$$\frac{\partial\, h_{\mp}(\rho,\theta)}{d\rho} = \frac{1}{\rho}\left\{\log\left(1 - \rho^{-1}e^{-\theta\sqrt{-1}}\right) + \log\left(1 - \rho e^{\theta\sqrt{-1}}\right) \mp 2\pi\sqrt{-1}\right\}$$

and

$$\frac{d\,\theta(\rho)}{d\rho} = \frac{1 - \rho^{-2}}{2\sin\theta(\rho)}$$

we have

$$\frac{d\, h_{\mp}(\rho)}{d\rho} = \left\{\frac{1}{\rho} + \frac{\sqrt{-1}(1 - \rho^{-2})}{2\sin\theta(\rho)}\right\}$$
$$\times \left\{\log\left(\left(1 - \rho^{-1}e^{-\theta(\rho)\sqrt{-1}}\right)\left(1 - \rho e^{\theta(\rho)\sqrt{-1}}\right)\right) \mp 2\pi\sqrt{-1}\right\}.$$

Therefore

$$\frac{d\, \mathrm{Im}\, h_{\mp}(\rho)}{d\rho} = \frac{1}{\rho}\left\{\arg\left(\left(1 - \rho^{-1}e^{-\theta(\rho)\sqrt{-1}}\right)\left(1 - \rho e^{\theta(\rho)\sqrt{-1}}\right)\right) \mp 2\pi\right\}$$
$$+ \frac{1 - \rho^{-2}}{2\sin\theta(\rho)}\left\{\log\left|\left(1 - \rho^{-1}e^{-\theta(\rho)\sqrt{-1}}\right)\left(1 - \rho e^{\theta(\rho)\sqrt{-1}}\right)\right|\right\}$$
$$= \frac{1}{\rho}\left\{\arg\left(\left(1 - \rho^{-1}e^{-\theta(\rho)\sqrt{-1}}\right)\left(1 - \rho e^{\theta(\rho)\sqrt{-1}}\right)\right) \mp 2\pi\right\}$$

since

$$\left|\left(1 - \rho^{-1}e^{-\theta(\rho)\sqrt{-1}}\right)\left(1 - \rho e^{\theta(\rho)\sqrt{-1}}\right)\right| = 1.$$

Thus we have

$$\begin{cases} \dfrac{d\,\operatorname{Im}h_+(\rho)}{d\rho} > 0, \\[2mm] \dfrac{d\,\operatorname{Im}h_-(\rho)}{d\rho} < 0 \end{cases}$$

and so $\operatorname{Im}h_\pm(\rho)$ on $D_\mp$ takes maximum at $\rho = 1$.

Since $\theta(1) = -\pi/3$, $\operatorname{Im}h_\mp(1) = 0$. Therefore (3.6) holds when $r = 1$. $\qquad\square$

Now we consider the case when $r \in \mathbb{R}$. In this case $m^2 + m^{-2} = 2\cos(2\pi r)$ and so

$$y = \exp\!\left(-\alpha(r)\sqrt{-1}\right)$$

with $\alpha(r) := \arccos(\cos(2\pi r) - 1/2)$ $(0 \le \arccos x \le \pi)$. From (3.4)

$H(y, m^2)$

$$= \operatorname{Li}_2(y^{-1}m^{-2}) - \operatorname{Li}_2(ym^{-2}) + \left(\log(-y) + \pi\sqrt{-1}\right)\log m^2$$

$$= \operatorname{Li}_2\!\left(\exp\!\big(\alpha(r) - 2\pi r)\sqrt{-1}\big)\right) - \operatorname{Li}_2\!\left(\exp\!\big(-\alpha(r) - 2\pi r)\sqrt{-1}\big)\right)$$

$$\quad + 2\pi(r-1)(\alpha(r) - 2\pi)$$

$$= \frac{\pi^2}{6} - \frac{2\pi + \alpha(r) - 2\pi r}{2}\left(\pi - \frac{2\pi + \alpha(r) - 2\pi r}{2}\right) + 2\sqrt{-1}\,\text{Л}\!\left(\frac{\alpha(r) - 2\pi r}{2}\right)$$

$$\quad - \frac{\pi^2}{6} + \frac{4\pi - \alpha(r) - 2\pi r}{2}\left(\pi - \frac{4\pi - \alpha(r) - 2\pi r}{2}\right) - 2\sqrt{-1}\,\text{Л}\!\left(\frac{-\alpha(r) - 2\pi r}{2}\right)$$

$$\quad + 2\pi(r-1)(\alpha(r) - 2\pi)$$

$$= -2\pi^2(r-1) + 2\sqrt{-1}\left\{\text{Л}\!\big(\pi r + \alpha(r)/2\big) - \text{Л}\!\big(\pi r - \alpha(r)/2\big)\right\}.$$

On the other hand from [13, Theorem 1.2]

$$2\pi r\,\operatorname{Re}\lim_{N\to\infty}\frac{\log J_N\!\big(E; \exp(2\pi r\sqrt{-1}/N)\big)}{N}$$

$$= 2\left\{\text{Л}\!\big(\pi r + \alpha(r)/2\big) - \text{Л}\!\big(\pi r - \alpha(r)/2\big)\right\}$$

if $r \notin \mathbb{Q}$ and $5/6 < r < 7/6$, or $r = 1$. Moreover from its proof we see that the sign of $J_N\!\big(E; \exp(2\pi r\sqrt{-1})/N\big)$ for large N is $(-1)^{\lfloor N(1-r)/r\rfloor}$, where $\lfloor x\rfloor$ is the greatest integer that does not exceed x. So

$$2\pi r\,\operatorname{Im}\lim_{N\to\infty}\frac{\log J_N\!\big(E; \exp(2\pi r\sqrt{-1}/N)\big)}{N} = 2\pi r\,\operatorname{Im}\lim_{N\to\infty}\frac{\log\left((-1)^{\lfloor N(1-r)/r\rfloor}\right)}{N}$$

$$= 2\pi^2(1-r).$$

Therefore we have

$$2\pi r\sqrt{-1}\lim_{N\to\infty}\frac{\log J_N\!\big(E; \exp(2\pi r\sqrt{-1}/N)\big)}{N} = H(y, m^2)$$

if $5/6 < r < 7/6$ and $r \notin \mathbb{Q}\setminus\{0\}$.

Putting

$$u := 2\pi r\sqrt{-1} - 2\pi\sqrt{-1}$$

and

$$H(u) := H(y, m^2),$$

we have proved the following theorem.

Theorem 3.8. *Let E be the figure-eight knot. There exists a neighborhood U of 0 in $\mathbb{C}$ such that for any $u \in (U \setminus \pi\sqrt{-1}\mathbb{Q}) \cup \{0\}$,*

$$(u + 2\pi\sqrt{-1}) \lim_{N \to \infty} \frac{\log\left(J_N\left(E; \exp\left(\frac{u + 2\pi\sqrt{-1}}{N}\right)\right)\right)}{N} = H(u).$$

4. Proof of the main theorem

In this section we will prove the main theorem.

First we will calculate the derivative of $H(u) := H(y, m^2)$.

Lemma 4.1.
$$\frac{d\left(H(u) - \pi\sqrt{-1}u\right)}{du} = \frac{v}{2}.$$

Proof. We first calculate the partial derivatives of $H(\xi, \eta)$. We have

$$\frac{\partial H(\xi, \eta)}{\partial \xi} = \frac{1}{\xi}\left\{\log\left(1 - \xi^{-1}\eta^{-1}\right) + \log\left(1 - \xi\eta^{-1}\right) + \log\eta\right\}$$
$$= \frac{1}{\xi}\log\left(\eta + \eta^{-1} - \xi - \xi^{-1}\right)$$

and

$$\frac{\partial H(\xi, \eta)}{\partial \eta} = \frac{1}{\eta}\left\{\log\left(1 - \xi^{-1}\eta^{-1}\right) - \log\left(1 - \xi\eta^{-1}\right) + \log(-\xi)\right\}$$
$$= \frac{1}{\eta}\left\{\log\left(\frac{1 - \xi\eta}{\eta - \xi}\right) + \pi\sqrt{-1}\right\}.$$

Since

$$\left.\frac{\partial H(\xi, \eta)}{\partial \xi}\right|_{\xi = y, \eta = m^2} = \frac{1}{y}\log\left(m + m^{-1} - y - y^{-1}\right) = 0$$

from (3.2) we have

$$\frac{d\,H(u)}{du} = \left.\frac{\partial H(\xi, \eta)}{\partial \xi}\right|_{\xi = y, \eta = m^2} \times \frac{d\,y}{du} + \left.\frac{\partial H(\xi, \eta)}{\partial \eta}\right|_{\xi = y, \eta = m^2} \times \frac{d\,m^2}{du}$$
$$= \log\left(\frac{1 - ym^2}{m^2 - y}\right) + \pi\sqrt{-1}.$$

But since

$$\begin{cases} ym^2 + zm^2 = 1 \\ z^{-1} + m^2 y^{-1} = 1, \end{cases}$$

we have

$$\frac{d\left(H(u) - \pi\sqrt{-1}u\right)}{du} = \log\left(z(1 - z)\right) = \frac{v}{2}.$$

$\square$

Next we will calculate $H(0)$.

Lemma 4.2.
$$H(0) = \sqrt{-1}\,\mathrm{Vol}\left(S^3 \setminus E\right)$$

Proof. Since $m\big|_{u=0} = -1$ and $y\big|_{u=0} = \exp(-\pi\sqrt{-1}/3)$, we have

$$H(0) = \mathrm{Li}_2\left(e^{-\pi\sqrt{-1}/3}\right) - \mathrm{Li}_2\left(e^{\pi\sqrt{-1}/3}\right) = 4\sqrt{-1}\,\text{Л}(\pi/6)$$

from (3.3).

On the other hand from the equality

$$\text{Л}(2\theta) = 2\,\text{Л}(\theta) - 2\,\text{Л}(\pi/2 - \theta)$$

(see for example [9, Lemma 1]), we have $4\,\text{Л}(\pi/6) = 6\,\text{Л}(\pi/3)$, which equals the volume of $S^3 \setminus E$. □

Proof of Corollary 2.4. We see that

$$\Phi(u) = 4H(u) - 4\pi\sqrt{-1}u - 4H(0)$$

and so

$$f(u) = H(u) - \pi\sqrt{-1}u - uv/4 - \sqrt{-1}\,\text{Vol}\left(S^3 \setminus E\right)$$

since $\text{CS}\left(S^3 \setminus E\right) = 0$. Therefore from Theorem 2.1

$$\text{Vol}\left(E_u\right) + \sqrt{-1}\,\text{CS}\left(E_u\right) = \frac{H(u)}{\sqrt{-1}} - \pi u - \frac{uv}{4\sqrt{-1}} - \frac{\pi}{2}\lambda(\gamma).$$

□

Appendix. Optimistic limit

In this appendix, we will consider integral surgeries along the figure-eight knot.

Let $E_{p,1}$ be the p-surgery along the figure-eight knot with $p \in \mathbb{Z}$. Then our parameter u and v satisfies $pu + v = 2\pi\sqrt{-1}$. From Corollary 2.4 we have

$$\text{Vol}(E_{p,1}) + \sqrt{-1}\,\text{CS}(E_{p,1}) \equiv \frac{H(u)}{\sqrt{-1}} - \pi u - \frac{uv}{4\sqrt{-1}} - \frac{\pi}{2}\lambda(\gamma) \mod \pi^2\sqrt{-1}\mathbb{Z}.$$

By [8, (4.6)], we have

$$\lambda(\gamma) = \frac{2\pi\sqrt{-1}}{p} - \frac{v}{p} = u.$$

(Note that we follow orientation conventions of [16, 24] and so the sign of v is opposite to that in [8].) Therefore

$$\text{Vol}(E_{p,1}) + \sqrt{-1}\,\text{CS}(E_{p,1})$$
$$\equiv -\sqrt{-1}\,\text{Li}_2\left(y^{-1}\exp(-u)\right) + \sqrt{-1}\,\text{Li}_2\left(y\exp(-u)\right) - \sqrt{-1}u\left\{\log(-y) + \pi\sqrt{-1}\right\}$$
$$- \pi u - \frac{u}{4\sqrt{-1}}(2\pi\sqrt{-1} - pu) - \frac{\pi}{2}u$$
$$\equiv \frac{1}{\sqrt{-1}}\left\{\text{Li}_2\left(y^{-1}\exp(-u)\right) - \text{Li}_2\left(y\exp(-u)\right) + u\log(-y) - u\pi\sqrt{-1} + \frac{pu^2}{4}\right\}$$
$$\mod \pi^2\sqrt{-1}\mathbb{Z}$$

and so

$$- \text{CS}(E_{p,1}) + \sqrt{-1}\,\text{Vol}(E_{p,1})$$
$$\equiv \text{Li}_2\left(y^{-1}\exp(-u)\right) - \text{Li}_2\left(y\exp(-u)\right) + u\log(-y) - u\pi\sqrt{-1} + \frac{pu^2}{4} \mod \pi^2\mathbb{Z}.$$

Now we define

$$V_p(\xi, \eta) := \text{Li}_2\left(\xi^{-1}\eta^{-1}\right) - \text{Li}_2\left(\xi\eta^{-1}\right) + (\log(-\xi))(\log\eta) - \pi\sqrt{-1}\log\eta + \frac{p}{4}(\log\eta)^2$$
$$= H(\xi, \eta) + \frac{p}{4}(\log\eta)^2 - 2\pi\sqrt{-1}\log\eta.$$

We will calculate $\partial V_p(\xi, \eta)/\partial\xi\big|_{\xi=y,\eta=m^2}$ and $\partial V_p(\xi, \eta)/\partial\eta\big|_{\xi=y,\eta=m^2}$. First we have from (3.2)

$$\frac{\partial V_p(\xi, \eta)}{\partial\xi}\bigg|_{\xi=y,\eta=m^2} = \frac{\partial H(\xi, \eta)}{\partial\xi}\bigg|_{\xi=y,\eta=m^2} = 0.$$

Next we have

$$\frac{\partial V_p(\xi, \eta)}{\partial\eta}\bigg|_{\xi=y,\eta=m^2} = \frac{\partial H(\xi, \eta)}{\partial\eta}\bigg|_{\xi=y,\eta=m^2} + \frac{p\log\left(m^2\right)}{2m^2} - \frac{2\pi\sqrt{-1}}{m^2}.$$

On the other hand

$$
\begin{aligned}
\frac{v}{2} &= \frac{d\,H(u,m^2)}{du} - \pi\sqrt{-1} \\
&= \left.\frac{\partial\,H(\xi,\eta))}{d\xi}\right|_{\xi=y,\eta=m^2} + \left.\frac{\partial\,H(\xi,\eta)}{d\eta}\right|_{\xi=y,\eta=m^2} \times \frac{d\,m^2}{du} - \pi\sqrt{-1} \\
&= \exp u \times \left.\frac{\partial\,H(\xi,\eta)}{d\eta}\right|_{\xi=y,\eta=m^2} - \pi\sqrt{-1}.
\end{aligned}
$$

Therefore

$$
\left.\frac{\partial\,V_p(\xi,\eta)}{\partial\eta}\right|_{\xi=y,\eta=m^2} = \frac{v/2 + \pi\sqrt{-1}}{\exp u} + \frac{pu}{2\exp u} - \frac{2\pi\sqrt{-1}}{\exp u} = 0.
$$

In [11] the first author calculated a 'fake' limit of the Witten–Reshetikhin–Turaev invariants of the $(p,1)$-Dehn surgery of the figure-eight knot and obtained the following observation.

Observation. *Let p be an integer between -100 and 100, and τ_p the 'optimistic' limit of the Witten–Reshetikhin–Turaev invariant of the $(p,1)$-Dehn surgery along the figure-eight knot. Then there exists (ξ_0,η_0) such that the following equalities hold numerically (up to 8 digits):*

(1) $\dfrac{\partial V_p}{\partial \xi}(\xi_0,\eta_0) = \dfrac{\partial V_p}{\partial \eta}(\xi_0,\eta_0) = 0$, *and*

(2) $\tau_p = V_p(\xi_0,\eta_0) = \mathrm{CS}(E_{p,1}) + \sqrt{-1}\,\mathrm{Vol}(E_{p,1})$.

Note that the observation above was confirmed by the second author [22]. Note also that the sign of CS is reversed since the orientation convention here is different.

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